Indian J. Phys. **47**, 295-300 (1973)

Radial flow of heat in two concentric spheres

Debadideb Bhattacharya, S. K. Ghosh and P K. Sen

Department of Physics, Jadavpur University, Calcutta-32

(*Received* 24 *April* 1972, *revised* 13 *June* 1972 *and* 1 *September* 1972)

The problem of radial flow of heat in two concentric spheres has been solved in this paper and the temperatures in different regions of the spheres have been obtained. Operational method has been used to solve the basic equations. The problem is discussed for the particular cases when

i) the difference of radii is not small,
ii) the difference of radii is small.

Introduction

In solving the general problem of heat flow in two composite spheres we make the following assumptions :

1) The surface and initial conditions are such that the isothermal surfaces are concentric spheres,

2) The media in the two regions are isotropic as regards conductivity, density and specific heat,

3) We neglect the loss of heat in our calculations,

4) There is no contact resistance at $r = a$.

Symbols Used

We write v_1, k_1, ρ_1, c_1 and h_1 for the temperature, conductivity, density, specific heat and diffusivity respectively, in the outer shell ($a < r < b$) and in the inner core ($0 < r < a$) the corresponding quantities are v_2, k_2, ρ_2, c_2, h_2
Let $l = b-a$, the difference of the radii of two spheres,

r, the variable co-ordinate,

V, the temperature of the sphere at $r = b$,

t, variable time,

$D = \frac{d}{dt}$ the operator

Mathematical Solution of the Problem

In the case of heat flow in the spheres when the initial and surface conditions are such that the isothermal surfaces are concentric spheres and the temperature thus depends only on r and t, the equations to be solved are

$$\frac{\partial v_1}{\partial t} = h_1\left(\frac{\partial^2 v_1}{\partial r^2}+\frac{2}{r}\frac{\partial v_1}{\partial r}\right), \quad a < r < b, t > 0 \qquad \dots \quad (1)$$

$$\frac{\partial v_2}{\partial t} = h_2\left(\frac{\partial^2 v_2}{\partial r^2}+\frac{2}{r}\frac{\partial v_2}{\partial r}\right), \quad 0 < r < a,\ t > 0 \qquad \dots \quad (2)$$

Writing $u_1 = v_1 r$ and $u_2 = v_2 r$, the equations (1) and (2) reduce to

$$\frac{\partial^2 u_1}{\partial r^2} - \frac{1}{h_1}\cdot\frac{\partial u_1}{\partial t} = 0, \quad a < r < b;\ t > 0 \qquad \dots \quad (3)$$

$$\frac{\partial^2 u_2}{\partial r^2} - \frac{1}{h_2}\cdot\frac{\partial u_2}{\partial t} = 0, \quad o < r < a,\ t > 0. \qquad \dots \quad (4)$$

For the composite spheres, with initial temperature zero and $r = b$, kept at constant temperature V for $t > 0$, the equations (3) and (4) in operational form reduce to

$$\frac{\partial^2 u_1}{\partial r^2} - q_1{}^2 u_1 = 0 \qquad \dots \quad (5)$$

$$\frac{\partial^2 u_2}{\partial r^2} - q_2{}^2 u_2 = 0 \qquad \dots \quad (6)$$

where $$q_1{}^2 = \frac{D}{h_1} \text{ and } q_2{}^2 = \frac{D}{h_2}.$$

As the rates of heat conduction at $r = a$ are equal from both the sides, the boundary conditions are

$$k_1\frac{\partial v_1}{\partial r} = k_2\frac{\partial v_2}{\partial r} \quad \text{at} \quad r = a,\ t > 0 \qquad \dots \quad (7)$$

$$v_1 = v_2 \quad \text{at} \quad r = a, \quad t > 0. \qquad \dots \quad (8)$$

The solutions of equations (5) and (6) are given by

$$u_1 = A_1 \cosh q_1 r + B_1 \sinh q_1 r \qquad \dots \quad (9)$$

$$u_{22} = c_1 \cosh q_2 r + D_1 \sinh q_2 r \qquad \dots \quad (10)$$

where A_1, B_1, C_1 and D_1 are arbitrary constants.

Evaluating the arbitrary constants from equations (7) and (8), the equations (9) and (10) turn out to be,

$$v_1 = \frac{u_1}{r} = \frac{bV}{r} \; \frac{\cosh q_1(r-a)+\sigma \sinh q_1(r-a)}{\cosh q_1 l+\sigma \sinh q_1 l} \quad \ldots \quad (11)$$

$$v_2 = \frac{u_2}{r} = \frac{bV}{r} \; \frac{\sinh q_2 r}{\sinh q_2 a(\cosh q_1 l+\sigma \sinh q_1 l)} \quad \ldots \quad (12)$$

where $$\sigma = \frac{aq_2k_2 \coth (q_2a)-(k_2-k_1)}{aq_1k_1}. \quad \ldots \quad (13)$$

Now two cases will be discussed :

Case—I

When a is large $\coth(q_2a) \to 1$ then σ from (13) becomes

$$\sigma = \sigma' = \left(\frac{k_2\rho_2c_2}{k_1\rho_1c_1} \right)^{\frac{1}{2}} - \frac{k_2-k_1}{aq_1k_1}.$$

Neglecting $\frac{k_2-k_1}{aq_1k_1}$ which is small, $\sigma = \sigma' = \left(\frac{k_2\rho_2c_2}{k_1\rho_1c_1} \right)^{\frac{1}{2}}$ is a constant. So $m = \frac{\sigma-1}{\sigma+1}$ is also a constant.

Then operational solutions of equations (11) and (12) turn out as

$$v_1 = \frac{bV}{r} \cdot \sum_{n=0}^{\infty} (m)^n \, [\operatorname{erfc}(\{(2n+1)l-(r-a)\}/2(h_1t)^{\frac{1}{2}})-m \operatorname{erfc} (\{(2n+1)l+(r-a)\}/2(h_1t)^{\frac{1}{2}})] \quad (14)$$

$$v_2 = \frac{2bV}{r(1+\sigma)} \sum_{n=0}^{\infty} (m)^n \operatorname{erfc} (\{(2n+1)l+h(a-r)\}/2(ht)^{\frac{1}{2}}). \quad (15)$$

The temperature gradient at the outer shell is found to be

$$\left(\frac{\partial v_1}{\partial r} \right)_{r=b} = -\frac{V}{b}+Vq_1\left(1+2 \sum_{n=1}^{\infty} (m)^n \exp \, (-2nq_1l)\right) \quad (16)$$

or $$\left(\frac{\partial v_1}{\partial r} \right)_{r=b} = -\frac{V}{b}+ \frac{V}{(\pi h_1 t)^{\frac{1}{2}}} \left(1+2 \sum_{n=1}^{\infty} (m)^n \; \exp \left(- \frac{n^2l^2}{h_1t} \right)\right). \quad (17)$$

For large values of time the exponential in equation (17) may be replaced by unity and we have

$$\left(\frac{\partial v_1}{\partial r} \right)_{r=b} = -\frac{V}{b}+ \frac{V}{(\pi h_1 t)^{\frac{1}{2}}} \; [1+2m(1+m+m^2+\ldots)]$$

$$= -\frac{V}{b}+\frac{V}{(\pi h_1 t)^{\frac{1}{2}}} \left(\frac{k_2\rho_2c_2}{k_1\rho_1c_1} \right)^{\frac{1}{2}} . \quad \ldots \quad (18)$$

The same result of temperature gradient is obtained by Carslaw & Jaeger (1959) in composite solids and by Ghosh & Bhattacharya (1970) in the case of linear flow of heat in semi-infinite finite solid, other than the correction term $-V/b$ which is very small.

Case—II

When a is large and l is small *i,e.*, when a thin spherical shell of some material is attached with it, expanding and retaining upto the first power of l, we have from equation (12)

$$\frac{v_2}{V} = \frac{b}{r} \cdot \frac{\exp\{-q_2(a-r)\}}{1+\sigma l q_1}$$

$$\frac{b}{r}\, w\, \frac{\exp\{-q_2(a-r)\}}{q_2+w}. \qquad \ldots \quad (19)$$

where $w = \frac{k_1}{k_2 l}$ a constant.

The operational solution of equation (19) is given by

$$\frac{v_2}{V} = \frac{b}{r}\,[\text{erfc}\,\{(a-r)/2(h_2t)^{\frac{1}{2}}\{-\exp\{w(a-r)+h_2tw^2\}$$

$$\times\,\text{erfc}\{(a-r)/2(h_2t)^{\frac{1}{2}}+w(h_2t)^{\frac{1}{2}}\}]. \qquad \ldots \quad (20)$$

The equation (20) is computed by using the following data : The material of the spherical film is cork of conductivity $k_1 = 0.0001$ and diffusivity $h_1 = 0.0014$. The second material is taken to be copper whose conductivity $k_2 = 0.93$ and diffusivity $h_2 = 1.14$. The temperature v_2 is calculated in the specimen at different distances of $(a-r) = 10$ and 100 cms after one hour when the temperature is assumed to be steady and the value of $a = 10^8$ cms

Two theoretical graphs are drawn from equation (20) : (*i*) Thickness of the spherical shell *vs* temperature in figure 1; (*ii*) Distance *vs* temperature in figure 2.

The nature of the graph in figure 1 is the same as obtained by Ghosh & Bhattacharya (1970). It is evident from the figure 1 that at a given value of $(a-r)$ increasing the film thickness decreases the value of temperature. At low value of thickness, the temperature rapidly falls to a lower value.

From figure 2 it is evident that at large shell thickness the temperature in the inner core is affected slightly but at small shell thickness the temperature distribution in the inner core is very prominent. But for large value of $(a-r)$ *i.e.*, in the neighbourhood of the central core the temperature distribution is only slightly affected by shell thickness.

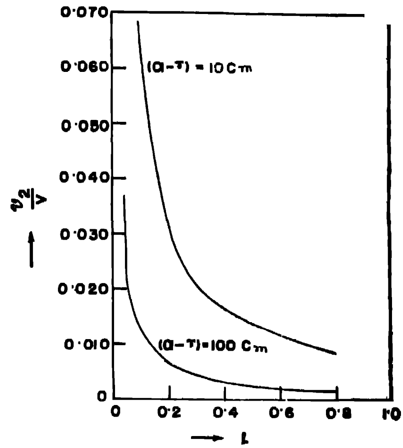

Figure 1. The thickness of the spherical shell l in cm is plotted against the temperature v_2/V for different values of $(a-r)$, e.g. $(a-r) = 10$ cm. and $(a-r) = 100$ cm.

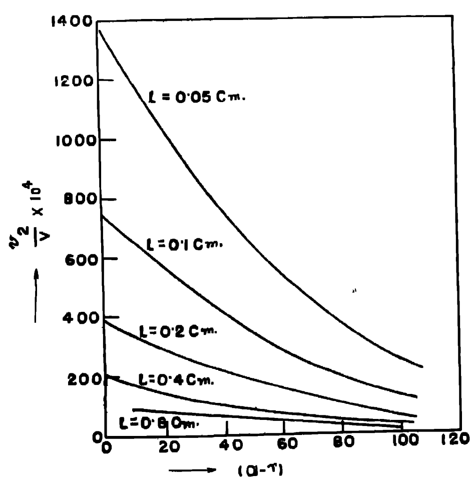

Figure 2. The distance $(a-r)$ in cm is plotted against the temperature v_2/V for different values of l, *e.g.*, $l = 0.05$ cm, $l = 0.1$ cm, $l = 0.2$ cm, $l = 0.4$ cm. and $l = 0.8$ cm.

The equation (20) gives approximately the temperature at any depth in the region $0 < x < a$ bounded either by a thin spherical film or a film of finite thickness having definite thermal capacity.

References

Carslaw H. S. & Jaeger J. C. 1959 *Conduction of Heat in Solids* (Oxford Publication), 2nd. Ed. pp. 319-29, pp. 351-2.

Ghosh S. K. & Bhattacharya D. 1970 *Indian J. Phys.* **44**, 101.

Jeffreys H & Jeffreys B. 1966 *Methods of Mathematical Physics* (Cambridge), 3rd Ed. pp. 228-43.